Lax Pairs for Additive Difference Painlevé Equations

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ABSTRACT

A Lax pair for the additive difference Painlevé equation of type $E_7^{(1)}$ is explicitly obtained as certain linear difference equations of scalar form. The compatibility of the Lax pair is given by using certain characterization of the coefficients in the Lax equation. Some Lax pairs for types $E_6^{(1)}$, $D_4^{(1)}$, and $A_3^{(1)}$ are also given by the degeneration.

KEY WORDS: Lax pair, additive difference Painlevé equation, Padé method, Padé interpolation, birational transformation

1 Introduction

The second order discrete Painlevé equations were classified in [23] based on rational surfaces connected to extended affine Weyl groups. There exist three types of discrete Painlevé equations in the classification: elliptic difference (e-), multiplicative difference (q-) and additive difference (d-) types. For each difference type, the list of types of affine Weyl groups arising as the symmetry of Bäcklund transformations are given as Figure 1.

Here $A \to B$ means that B is obtained from A by degeneration.

The main subject of this paper is the d-$E_7^{(1)}$ equation, namely the d-Painlevé equation of type $E_7^{(1)}$ [6, 10, 25]. For variables (f, g) in $\mathbb{P}^1 \times \mathbb{P}^1$ and parameters $\delta, a_i, b_i, (i = 1, 2, 3, 4) \in \mathbb{C}^\times$ with a constraint $a_1 - a_2 - a_3 + a_4 + b_1 + b_2 - b_3 + b_4 + \delta = 0$, define a shift operator T as

$$T : \begin{pmatrix} a_1, & a_2, & a_3, & a_4 & \\ & & & & , f, g \\ b_1, & b_2, & b_3, & b_4 & \end{pmatrix} \mapsto \begin{pmatrix} a_1 - \delta, & a_2 - \delta, & a_3 - \delta, & a_4 - \delta & \\ & & & & , \overline{f}, \overline{g} \\ b_1, & b_2, & b_3, & b_4 & \end{pmatrix}. \tag{1}$$

Here for any object X the corresponding shifts are denoted as $\overline{X} := T(X)$ and $\underline{X} := T^{-1}(X)$. Then the d-$E_7^{(1)}$ equation can be described by the birational transformation $T^{-1}(g) = \underline{g}(f, g)$ and $T(f) = \overline{f}(f, g)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ as follows:

$$\frac{(f - g - v)(f - \underline{g} - v - \delta)}{(f - g)(f - \underline{g})} = \frac{A(f)}{B(f)}, \tag{2}$$

$$\frac{(f - g - v)(\overline{f} - g - v + \delta)}{(f - g)(\overline{f} - g)} = \frac{A(g + v)}{B(g)}, \tag{3}$$

where $A(x) := \prod_{i=1}^4 (x - a_i)$, $B(x) := \prod_{i=1}^4 (x - b_i)$, $v := a_1 + a_4 - b_3$.

Let us briefly recall a background on Lax pairs of discrete Painlevé equations. The 2×2 matrix Lax pairs for type q-$D_5^{(1)}$ in [9] was derived using the connection preserving deformation of a 2×2 matrix system of q-difference equations. Some 2×2 matrix Lax pairs for types from d-$D_4^{(1)}$ to d-$A_1^{(1)}$ were derived in [5], using a Schlesinger transformation of differential equations. The 2×2 matrix Lax pairs for type q-$E_6^{(1)}$ in [24] was derived, making use of a similar way as in[9]. Some 2×2 matrix Lax pairs were obtained utilizing moduli spaces of difference connection on $\mathbb{P}^1$ in [1] for types d-$E_6^{(1)}$ and d-$D_4^{(1)}$. The 2×2 matrix Lax pairs for types, from q-$A_4^{(1)}$ to q-$A_1^{(1)}$, in [24] were derived by utilizing a similar way as in [9, 24]. Certain matrix Lax pairs were obtained as certain Fuchsian system of differential equations

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in [2] for types d-$E_8^{(1)}$, d-$E_7^{(1)}$, and d-$E_6^{(1)}$. In [3, 4] the Lax pairs for types d-$E_7^{(1)}$, d-$E_6^{(1)}$, and d-$D_4^{(1)}$ have been constructed as reductions from so-called elementary Schlesinger transformations of some Fuchsian systems by computing more explicitly than in [2]. Some scalar Lax pairs of discrete Painlevé equations were given as linear difference equations, using a characterization in the coordinates $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$ for type e-$E_8^{(1)}$ in [27], and types q-$E_8^{(1)}$ and q-$E_7^{(1)}$ in [28]. In [10], the scalar Lax pairs for all the discrete Painlevé equations have recently been proposed by utilizing the characterization in the coordinates (f, g).

This paper is organized as follows: In Section 2.1, the Lax pair for type d-$E_7^{(1)}$ is explicitly obtained as certain linear difference equations of scalar form. In Section 2.2, we show that the d-$E_7^{(1)}$ equation (2) and (3) is the sufficient condition for the compatibility of the Lax pair by using certain characterization in terms of x (see Section 2.2), which are related but different from the characterization in the coordinates (f, g) (see Remark 2.2). In Section 3, the Lax pairs for types d-$E_6^{(1)}$, d-$D_4^{(1)}$ and d-$A_3^{(1)}$ are obtained by the degeneration. In Section 4, the relation between Padé interpolation problems and the results of this paper are discussed shortly as Concluding remarks 4.

2 The Lax Pair for d-Painlevé Equation of Type $E_7^{(1)}$

This section will consider the Lax pair for type d-$E_7^{(1)}$. In Section 2.1, we will give certain two linear difference equations as the scalar Lax pair for type d-$E_7^{(1)}$ and derive the d-$E_7^{(1)}$ equation. Section 2.2 will prove that equations (2) and (3) are sufficient conditions for the compatibility of the linear equations.

2.1 Lax Equations

Let us consider two linear equations for an unknown function $y(x)$: $L_2(x) = 0$ as the equation between $y(x)$, $y(x + \delta)$, $\overline{y}(x)$, and $L_3(x) = 0$ as the equation between $y(x)$, $\overline{y}(x)$, $\overline{y}(x - \delta)$, where $L_2(x)$ and $L_3(x)$ are given as linear three-term expressions

$$L_2(x) := (x - f)\overline{y}(x) - (x - g - v)y(x + \delta) + (x - g)y(x), \tag{4}$$

$$L_3(x) := w(x - \overline{f} - \delta)y(x) + A(x)(x - g - \delta)\overline{y}(x) - B(x - \delta)(x - g - v)\overline{y}(x - \delta), \tag{5}$$

and $f, g, \overline{f}, w$ are some variables depending on the parameters δ, a_i, and b_i but independent of x. Then we have

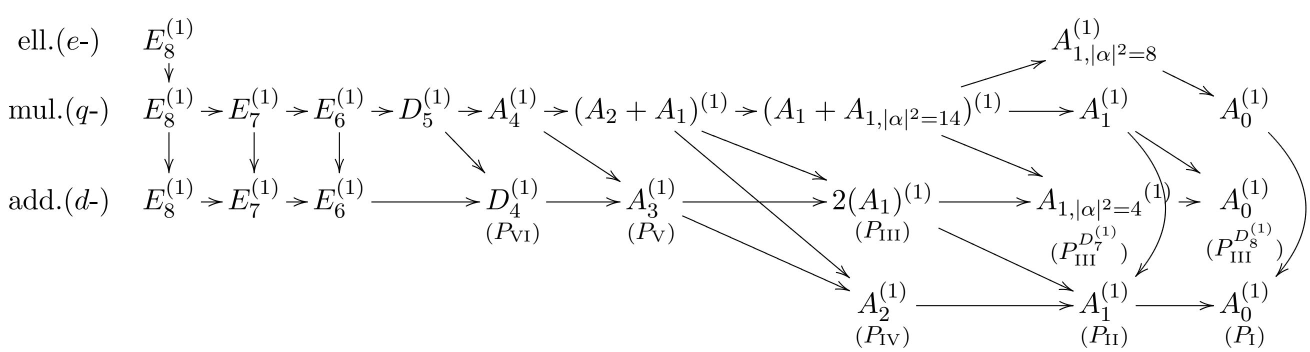

Figure 1: Degeneration diagram of affine Weyl group symmetries

Proposition 2.1. *The compatibility of the linear equations $L_2 = 0$ (4) and $L_3 = 0$ (5) gives conditions (2) and (3).*

Proof. Under the condition $x = f$, eliminating $y(x)$ and $y(x+\delta)$ from $L_2(x) = \underline{L}_3(x+\delta) = 0$ where

$$\begin{aligned}\underline{L}_3(x) = \underline{w}(x-f-\delta)\underline{y}(x) + A(x-\delta)(x-\underline{g}-\delta)y(x)\\ - B(x-\delta)(x-\underline{g}-v-2\delta)y(x-\delta),\end{aligned} \tag{6}$$

obtains equation (2). For $x = g$, eliminating $y(x+\delta)$ and $\overline{y}(x)$ from $L_2(x) = L_3(x+\delta) = 0$, we have the relation

$$w = \frac{v(v-\delta)B(g)}{(f-g)(\overline{f}-g)}. \tag{7}$$

In the case of $x = g+v$, eliminating $y(x)$ and $\overline{y}(x)$ from $L_2(x) = L_3(x) = 0$, we have the relation

$$w = \frac{v(v-\delta)A(g+v)}{(f-g-v)(\overline{f}-g-v+\delta)}. \tag{8}$$

Hence, eliminating w from relations (7) and (8), we obtain equation (3). □

Let us consider the three-term linear equation for the unknown function $y(x)$: $L_1(x) = 0$ as the equation between $y(x+\delta), y(x), y(x-\delta)$. Eliminating $\overline{y}(x)$ and $\overline{y}(x-\delta)$ from $L_2(x) = L_2(x-\delta) = L_3(x) = 0$ (4), (5), one has the equation $L_1(x) = 0$, where $L_1(x)$ is given as a linear three-term expression

$$\begin{aligned}L_1(x) := \frac{A(x)}{x-f}y(x+\delta) + \frac{B(x-\delta)}{x-f-\delta}y(x-\delta)\\ -\frac{1}{x-g-v}\Big[\frac{A(x)(x-g)}{x-f} + \frac{V(x-\delta)}{(x-f-\delta)(x-g-\delta)}\Big]\\ \times y(x),\end{aligned} \tag{9}$$

$$V(x) := B(x)(x-g-v)(x-g-v+\delta) - w(x-f)(x-\overline{f}). \tag{10}$$

Eliminating $\overline{f}$ and w from the equation $L_1 = 0$ (9) by using equation (3) and relation (7), the expression L_1 can be rewritten as

$$\begin{aligned}L_1(x) = \frac{A(x)}{x-f}\left[y(x+\delta) - \frac{x-g}{x-g-v}y(x)\right]\\ +\frac{B(x-\delta)}{x-f-\delta}\left[y(x-\delta) - \frac{x-g-v-\delta}{x-g-\delta}y(x)\right]\\ +v\left[\frac{B(g)}{(f-g)(x-g-\delta)} - \frac{A(g+v)}{(f-g-v)(x-g-v)}\right]\\ \times y(x).\end{aligned} \tag{11}$$

The linear difference equations $L_1 = 0$ (11) and $L_2 = 0$ (4) can be regarded as the Lax pair for type *d*-$E_7^{(1)}$, and in Section 2.2 it will be proved that the *d*-$E_7^{(1)}$ equation (2), (3) is the sufficient condition for the compatibility of $L_1 = 0$ and $L_2 = 0$. The equation $L_1 = 0$ is equivalent to the scalar Lax equation in [10] by using a suitable gauge transformation of $y(x)$. On the other hand, a 4×4 matrix Lax pair was given as a certain Fuchsian system of differential equations in [2]. In [4] certain 4×4 matrix Lax pair has been constructed as a reduction from two different kinds of elementary Schlesinger transformations by computing more explicitly than in [2].

Remark 2.2. *The linear equation $(f-g)(f-g-v)(x-f)(x-f-\delta)L_1(x) = 0$ (11) has the properties as a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to the coordinates $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$ as follows:*
(i) The expression $(f-g)(f-g-v)(x-f)(x-f-\delta)L_1(x)$ is a polynomial of bidegree $(3, 2)$ in (f, g).
(ii) As a polynomial, the expression $(f-g)(f-g-v)(x-f)(x-f-\delta)L_1$ vanishes at the the following 12 points $(f_i, g_i) \in \mathbb{P}^1 \times \mathbb{P}^1$ $(i = 1, \ldots, 12)$:

$$\begin{aligned}&(b_i, b_i)_{i=1}^4, (a_i, a_i - v)_{i=1}^4, (x, x),\\ &(x-\delta, x-v-\delta), (x, \tfrac{(x-v)y(x+\delta)-xy(x)}{y(x+\delta)-y(x)}),\\ &(x-\delta, \tfrac{(x-v-\delta)y(x)-(x-\delta)y(x-\delta)}{y(x)-y(x-\delta)}).\end{aligned} \tag{12}$$

Conversely, the polynomial $(f-g)(f-g-v)(x-f)(x-f-\delta)L_1$ is characterized by these properties up to multiplicative constant.

We note that similar properties of the equation $L_1 = 0$ for the discrete Painlevé equations have already been given in [10, 27, 28]. In Section 2.2, we gives different properties of the coefficients of $y(x+\delta), y(x)$, and $y(x-\delta)$ in terms of x.

2.2 Proof of the compatibility condition of the Lax pair

In Section 2.1, we derived the d-$E_7^{(1)}$ equation (2) and (3) as the necessary condition for the compatibility of the Lax pair (4) and (5) (or (4) and (11)). In this subsection, we will prove that the d-$E_7^{(1)}$ equation is the sufficient condition for the compatibility of the Lax pair.

Lemma 2.3. *The expression $(x-f)(x-f-\delta)L_1(x)$ (9) (or (11)) has the following properties:*

(i) It is a three-term linear expression among $y(x+\delta)$, $y(x)$, and $y(x-\delta)$, and the coefficients of these terms are polynomials of degree 5 in x.

(ii) The coefficients of $y(x+\delta)$ (resp. $y(x-\delta)$) has zero at $x=a_1,\dots,a_4$ (resp. $x=b_1+\delta,\dots,b_4+\delta$).

(iii) Under the condition

$$\frac{y(x+\delta)}{y(x)}=1+\frac{v}{x}+\frac{v(v-\delta)/2}{x^2}+\frac{c}{x^3}+O\Big(\frac{1}{x^4}\Big),$$
$$\frac{y(x-\delta)}{y(x)}=1-\frac{v}{x}+\frac{v(v-\delta)/2}{x^2}-\frac{c}{x^3}+O\Big(\frac{1}{x^4}\Big), \tag{13}$$

the terms $x^5,\dots,x^2$ in the expression $(x-f)(x-f-\delta)L_1(x)$ vanish, namely $(x-f)(x-f-\delta)L_1(x)=O(x^1)$ around $x=\infty$. Here $c\in\mathbb{C}$ is an arbitrary constant.

(iv) The equation $(x-f)(x-f-\delta)L_1=0$ has two apparent singularities $x=f,f+\delta$ (i.e., the solutions are regular there), where

$$\frac{y(f+\delta)}{y(f)}=\frac{f-g}{f-g-v}\quad for\ x=f,f+\delta \tag{14}$$

holds.

Conversely, the expression $(x-f)(x-f-\delta)L_1(x)$ is uniquely characterized by the properties (i) – (iv).

Proof. The property (i) is given by computation using equations (7) and (8). Specifically, the expression $\frac{V(x-\delta)}{x-g-\delta}$ in (9) reduces to a polynomial of degree 5 in x under condition (7). Furthermore, the coefficient of the term $y(x)$ is given as a polynomial of degree 5 in x using condition (8). The property (ii) is trivial. The property (iii) can easily be confirmed by substituting condition (13) into the equation $(x-f)(x-f-\delta)L_1(x)=0$ (11) [1]. The property (iv) follows by substituting $x=f,f+\delta$ into the equation $(x-f)(x-f-\delta)L_1(x)=0$ (11). The converse can be easily be confirmed by counting the number of free coefficients. □

We remark that two points $x=f,f+\delta$ are apparent singularities in the sense that the equation $(x-f)(x-f-\delta)L_1(x)=0$ is satisfied at those two points by the same condition (in this case (14)).

Eliminating $y(x)$ and $y(x+\delta)$ from $L_2(x)=L_3(x)=L_3(x+\delta)=0$ in (4) and (5) result in a three-term linear equation $L_1^*(x)=0$ for $\overline{y}(x+\delta),\overline{y}(x)$, and $\overline{y}(x-\delta)$ where

$$\begin{aligned}L_1^*(x):=&\frac{\overline{A}(x)}{x-\overline{f}}\overline{y}(x+\delta)+\frac{\overline{B}(x-\delta)}{x-\overline{f}-\delta}\overline{y}(x-\delta)\\&-\frac{1}{x-g-v}\Big[\frac{A(x)(x-g-\delta)}{x-\overline{f}-\delta}+\frac{V(x)}{(x-\overline{f})(x-g)}\Big]\\&\times\overline{y}(x).\end{aligned}\tag{15}$$

The following Lemma (and its proof) is similar to Lemma 2.3.

Lemma 2.4. *The expression $(x-\overline{f})(x-\overline{f}-\delta)L_1^*(x)$ (15) has the following properties:*

(i) It is a three-term linear expression for $\overline{y}(x+\delta)$, $\overline{y}(x)$, and $\overline{y}(x-\delta)$, and the coefficients of these terms are polynomials of degree 5 in x.

(ii) The coefficients of $\overline{y}(x+\delta)$ (resp. $\overline{y}(x-\delta)$) has zero at $x=a_1-\delta,\dots,a_4-\delta$ (resp. $x=b_1+\delta,\dots,b_4+\delta$).

(iii) Under the condition

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[1]
$$\begin{aligned}&(x-f)(x-f-\delta)L_1(x)=(x-f-\delta)A(x)\times\\&\Big[1+\frac{v}{x}+\frac{v(v-\delta)}{x^2}+\frac{c}{x}+O\Big(\frac{1}{x^4}\Big)-\frac{x-g}{x-g-v}\Big]y(x)\\&+(x-f)B(x-\delta)\times\\&\Big[1-\frac{v}{x}+\frac{v(v-\delta)/2}{x^2}-\frac{c}{x^3}+O\Big(\frac{1}{x^4}\Big)-\frac{x-g-v-\delta}{x-g-\delta}\Big]\\&\times y(x)+v(x-f)(x-f-\delta)\times\\&\Big[\frac{B(g)}{(f-g)(x-g-\delta)}-\frac{A(g+v)}{(f-g-v)\,(x-g-v)}\Big]y(x).\end{aligned}$$

$$\frac{\overline{y}(x+\delta)}{\overline{y}(x)} = 1 + \frac{\overline{v}}{x} + \frac{\overline{v}(\overline{v}-\delta)/2}{x^2} + \frac{\overline{c}}{x^3} + O\Big(\frac{1}{x^4}\Big),$$
$$\frac{\overline{y}(x-\delta)}{\overline{y}(x)} = 1 - \frac{\overline{v}}{x} + \frac{\overline{v}(\overline{v}-\delta)/2}{x^2} - \frac{\overline{c}}{x^3} + O\Big(\frac{1}{x^4}\Big), \tag{16}$$

the terms $x^5, \ldots, x^2$ in the polynomial $(x-\overline{f})(x-\overline{f}-\delta)L_1(x)$ vanish, namely $(x-\overline{f})(x-\overline{f}-\delta)L_1^*(x) = O(x^1)$ around $x = \infty$. Here $c \in \mathbb{C}$ is the same arbitrary constant as in (13).

(iv) The equation $(x-\overline{f})(x-\overline{f}-\delta)L_1^* = 0$ has two apparent singularities $x = \overline{f}, \overline{f}+\delta$, where

$$\frac{\overline{y}(\overline{f}+\delta)}{\overline{y}(\overline{f})} = \frac{\overline{B}(\overline{f})(\overline{f}-g-v+\delta)}{\overline{A}(\overline{f})(\overline{f}-g)} \quad for\ x = \overline{f}, \overline{f}+\delta \tag{17}$$

holds.

Conversely, the expression $(x-\overline{f})(x-\overline{f}-\delta)L_1^*(x)$ is uniquely characterized by the properties (i) – (iv).

The following is the main result of this paper.

Theorem 2.5. *The linear equations $L_1 = 0$ (11) and $L_2 = 0$ (4) for the unknown function $y(x)$ are compatible if and only if the d-$E_7^{(1)}$ equation (2) and (3) are satisfied.*

Proof. The compatibility means that the shift operator T changes the equation $L_1 = 0$ into the equation $L_1^* = 0$, i.e., the commutativity of the following:

$$\begin{array}{lcl} L_1^* = 0 \text{ (Lemma 2.4)} & \Leftrightarrow & L_1^* = 0 \text{ (15)} \\ \uparrow & & \uparrow \\ T\text{-shift (1)} & & L_2 = L_3 = 0 \text{ (4), (5)} \\ \uparrow & & \downarrow \\ L_1 = 0 \text{ (Lemma 2.3)} & \Leftrightarrow & L_1 = 0 \text{ (9)} \\ & \Leftrightarrow & L_1 = 0 \text{ (11).} \end{array}$$

This commutativity is almost clear from the characterizations (i) and (ii) of the equation $L_1 = 0$ (respectively $L_1^* = 0$) in Lemma 2.3 (respectively Lemma 2.4). The remaining task is to check that the operator T changes expression (14) into expression (17), utilizing the characterization (iii) of the equation $L_1 = 0$ (respectively $L_1^* = 0$) and equation (2). □

As the point of the proof, the following two are applied to type d-$E_7^{(1)}$ together: The first is that the equation $L_1(f, \overline{f}, g) = 0$ in terms of f, $\overline{f}$, and g is derived from the equations $L_2(f, g) = 0$ and $L_3(\overline{f}, g) = 0$ (see [10, 21, 27, 28]). The second is that the equation $L_1(f, \overline{f}, g) = 0$ is characterized as a polynomial in terms of x (see [19]).

3 Degenerations

In this section, we will consider degeneration limits from type d-$E_7^{(1)}$ to types d-$E_6^{(1)}$, d-$D_4^{(1)}$ and d-$A_3^{(1)}$.

3.1 Degeneration from type d-$E_7^{(1)}$ to type d-$E_6^{(1)}$

Degeneration from type d-$E_7^{(1)}$ to type d-$E_6^{(1)}$ is obtained by setting a transformation

$$a_4 \to \frac{-1}{\varepsilon},$$
$$a_3 \to \frac{-1}{\varepsilon} + a_1 - a_2 + b_1 + b_2 - b_3 + b_4 + \delta,$$
$$\frac{y(x+\delta)}{y(x)} \to \frac{\varepsilon y(x+\delta)}{y(x)}, \tag{18}$$

and taking the limit $\varepsilon \to 0$.
The time evolution is given by a shift operator

$$T : (a_1, a_2, b_1, b_2, b_3, b_4, f, g) \mapsto (a_1 - \delta, a_2 - \delta, b_1, b_2, b_3, b_4, \overline{f}, \overline{g}). \tag{19}$$

The d-Painlevé $E_6^{(1)}$ equation [10, 22, 25] is equivalent to a birational transformation

$$(f-g)(f-\underline{g}) = \frac{B(f)}{(f-a_1)(f-a_2)},$$
$$(f-g)(\overline{f}-g) = \frac{B(g)}{(g+u_1)(g+u_2)}, \tag{20}$$

where $B(x) := \prod_{i=1}^4 (x - b_i)$, $u_1 := a_2 - b_1 - b_2 - b_4 - \delta$, and $u_2 := a_1 - b_3$. The Lax pair is given as two linear difference equations

$$L_1(x) := \frac{(x-a_1)(x-a_2)}{x-f}\left[y(x+\delta)-(x-g)y(x)\right]$$
$$+\frac{B(x)}{x-f-\delta}\left[y(x-\delta)-\frac{y(x)}{x-g-\delta}\right]$$
$$+\left[(g+u_1)(g+u_2)-\frac{B(x)}{(f-g)(x-g-\delta)}\right]$$
$$\times y(x)=0,$$

$$L_2(x) := (x-f)\overline{y}(x)-y(x+\delta)+(x-g)y(x)=0. \tag{21}$$

The equation $L_1 = 0$ (21) is equivalent to the scalar Lax equation in [10] by using suitable gauge transformation of $y(x)$. We remark that a 2×2 matrix Lax pair was obtained utilizing moduli space of difference connection in [1], and a 3×3 matrix Lax pair was given as the Fuchsian system of differential equations in [2]. In [3, 4] certain 3×3 matrix Lax pair has been constructed as a reduction from elementary Schlesinger transformations by computing more explicitly than in [2].

3.2 Degeneration from type d-$E_6^{(1)}$ to type d-$D_4^{(1)}$

Degeneration from type d-$E_6^{(1)}$ to type d-$D_4^{(1)}$ is obtained by setting a transformation

$$\begin{gathered} b_3 \to -\frac{1}{\varepsilon},\ b_4 \to -\frac{1}{\varepsilon t},\ g \to -\frac{1}{g\varepsilon}, \\ \frac{y(x+\delta)}{y(x)} \to \frac{y(x+\delta)}{\varepsilon y(x)},\ \frac{\overline{y}(x)}{y(x)} \to \frac{\overline{y}(x)}{\varepsilon y(x)}, \end{gathered} \tag{22}$$

and taking the limit $\varepsilon \to 0$.
The time evolution is given by a shift operator

$$\begin{aligned} T &: (a_1,a_2,b_1,b_2,t,f,g) \\ &\mapsto (a_1-\delta,a_2-\delta,b_1,b_2,t,\overline{f},\overline{g}). \end{aligned} \tag{23}$$

The d-$D_4^{(1)}$ equation [5, 10, 22, 25] is equivalent to a birational transformation

$$\begin{gathered} g\underline{g} = \frac{t(f-a_1)(f-a_2)}{(f-b_1)(f-b_2)}, \\ f+\overline{f}-b_1-b_2+\frac{a_1}{g-1}+\frac{tu}{g-t}=0, \end{gathered} \tag{24}$$

where $u := a_2-b_1-b_2-\delta$. The Lax pair is given as two linear difference equations

$$L_1(x) := \frac{t(x-a_1)(x-a_2)}{x-f}\left[y(x+\delta)-\frac{y(x)}{g}\right]$$
$$+\frac{(x-b_1-\delta)(x-b_2-\delta)}{x-f-\delta}\left[y(x-\delta)-gy(x)\right]$$
$$+(g-1)(g-t)\left[\frac{a_1}{g-1}+\frac{x+f-a_1-a_2}{g}+\frac{u}{g-t}\right]$$
$$\times y(x)=0,$$

$$L_2(x) := (x-f)\overline{y}(x)-y(x+\delta)+\frac{y(x)}{g}=0. \tag{25}$$

The equation $L_1 = 0$ (25) is equivalent to the scalar Lax equation in [10] by using suitable gauge transformation of $y(x)$. We remark that a 2×2 matrix Lax pair for type d-$\mathrm{P_V}$ was obtained by utilizing a Schlesinger transformation of a differential equation in [5] and a 2×2 matrix Lax pair was obtained utilizing moduli space of difference connection in [1]. In [3] certain 2×2 matrix Lax pair has been constructed as a reduction from elementary Schlesinger transformations by explicit computation.

3.3 Degeneration from type d-$D_4^{(1)}$ to type d-$A_3^{(1)}$

Degeneration from type d-$D_4^{(1)}$ to type d-$A_3^{(1)}$ is obtained by setting a transformation

$$a_2 \to -\frac{1}{\varepsilon},\ t \to \varepsilon t, \tag{26}$$

and taking the limit $\varepsilon \to 0$.
The time evolution is given by a shift operator

$$T : (a_1,b_1,b_2,t,f,g) \mapsto (a_1-\delta,b_1,b_2,t,\overline{f},\overline{g}). \tag{27}$$

The d-$A_3^{(1)}$ equation [5, 10, 22, 25] is equivalent to a birational transformation

$$\begin{gathered} g\underline{g} = \frac{t(f-a_1)}{(f-b_1)(f-b_2)}, \\ f+\overline{f}-b_1-b_2-\frac{t}{g}+\frac{a_1}{g-1}=0. \end{gathered} \tag{28}$$

The Lax pair is given as two linear difference equations

$$L_1(x) := \frac{t(x-a_1)}{x-f}\left[y(x+\delta)-\frac{y(x)}{g}\right] + \frac{(x-b_1-\delta)(x-b_2-\delta)}{x-f-\delta}\left[y(x-\delta)-gy(x)\right] + (g-1)\left[x+f-b_1-b_2-\delta-\frac{t}{g}+\frac{a_1}{g-1}\right] \times y(x) = 0,$$

$$L_2(x) := (x-f)\overline{y}(x) - y(x+\delta) + \frac{y(x)}{g} = 0. \quad (29)$$

The equation $L_1 = 0$ (29) is equivalent to the scalar Lax equation in [10] by using suitable gauge transformation of $y(x)$. We remark that a 2×2 matrix Lax pair for type d-$\mathrm{P_{IV}}$ was obtained by using a Schlesinger transformation of a differential equation in [5].

4 Concluding remarks

As mentioned in Section 1, the results of this paper were obtained making use of Padé interpolation problems. We will discuss the relation with Padé problem and the results shortly.

There exists a simple method to study the Painlevé/Garnier equations by using Padé approximation [26]. In this method, one can obtain the evolution equation, the Lax pair and some special solutions simultaneously, starting from a suitable Padé approximation (or interpolation) problem. Concretely, for a suitable given function $Y(x)$, we look for polynomials $P_m(x)$ and $Q_n(x)$ of degree m and $n \in \mathbb{Z}_{\geq 0}$, satisfying the approximation condition:

$$Y(x) \equiv \frac{P_m(x)}{Q_n(x)} \quad (\mathrm{mod}\ x^{m+n+1}), \quad (30)$$

or the interpolation condition:

$$Y(x_s) = \frac{P_m(x_s)}{Q_n(x_s)} \quad (s = 0, \ldots, m+n). \quad (31)$$

Define certain shift operator T (e.g.(1)) and consider two three-term linear relations satisfied by the function $y(x) = P_m(x), Y(x)Q_n(x)$, such as the equations $L_2(x) = L_3(x) = 0$ (e.g.(4), (5)). The compatibility of the linear relations gives Painlevé equation (e.g.(2), (3)). Moreover special solutions of the Painlevé equation can be obtained from the polynomials $P_m(x)$ and $Q_n(x)$ of the Padé condition (30) (or (31)). We call the method mentioned above "*Padé method*". The Padé method based on (30) and (31) have been applied in [7, 16, 19, 26, 20] and [7, 15, 17, 18, 19, 21, 29, 30, 20] respectively. For the works related to Padé approximation, see [8, 11, 12, 13].

Recently the results of this paper have been obtained in [17] by the Padé interpolation problems (31) on the additive (δ-) grid $x_s = s\delta$. The interpolated sequences $Y_s := Y(x_s)$ can be chosen as follows:

	d-$E_7^{(1)}$	d-$E_6^{(1)}$	d-$D_4^{(1)}$	d-$A_3^{(1)}$
Y_s	$\prod_{i=1}^{3}\frac{(\frac{b_i}{\delta})_s}{(\frac{a_i}{\delta})_s}$	$\prod_{i=1}^{2}\frac{(\frac{b_i}{\delta})_s}{(\frac{a_i}{\delta})_s}$	$c^s\frac{(\frac{b_1}{\delta})_s}{(\frac{a_1}{\delta})_s}$	$d^s(\frac{b_1}{\delta})_s$

(32)

Here $a_i, b_i, c, d \in \mathbb{C}^\times$ are parameters, and $a_1+a_2+a_3-b_1-b_2-b_3-\delta(m-n)=0$ is a constraint for the parameters in type d-$E_7^{(1)}$, and Pochhammer's symbol is defined by

$$(a_1, a_2, \cdots, a_i)_j := \prod_{k=0}^{j-1}(a_1+k)(a_2+k)\cdots(a_i+k). \quad (33)$$

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